A playful approach to combinatorics and algbra via permutation puzzles

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A twofold problem in math education

- “Discrete math” (combinatorics, graphs etc.) in the middle and high school curriculum: Underrepresented. “Continuous” structures comparatively dominating (e.g., understanding of “functions”). Nevertheless, increasing importance of discrete structures in the mathematics of the AI era.
- Consequence: University students often feel alienated when first meeting “higher” discrete math (e.g. algebra), and may question the relevance of the subject for their vocational goal.

Goals

On the students' side:

- "Playful" introduction into (elementary, but not entirely trivial aspects of) discrete math topics related to the theory of permutations.
- Perception of functions as "processes" between different states, "progress" in a problem, rather than purely analytic or algebraic function concept.
- Concrete problem-solving activities with "try and error" leading to conceptual insights.
- **Not among the goals**: Explicit introduction of higher abstract algebra into school classrooms!

On the teachers' side:

- Demonstration of the applicability of "abstract" topics familiar from university math courses, via concrete problems.
- Possibility to "re-discover" algebraic facts from a different perspective.

Game 1: "Swap"

Let's play the following little game.

- Take an arbitrary amount (not too large) of pieces numbered 1, 2, ..., n; shuffle them arbitrarily and place them in a row.

$$④ \quad ② \quad ⑤ \quad ③ \quad ⑦ \quad ⑥ \quad ①$$

- A legal "move" consists of picking any two pieces lying directly next to each other, and swapping them.

Game 1: "Swap"

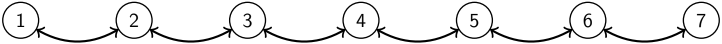


Figure: Legal moves in "Swap".

Activity (Individual or group)

- **Question 1**: Via performing suitable moves, is it possible to bring the puzzle into the "ordered" sequence $1, 2, \ldots, n$?
- **Question 2**: Can you give a general description of how to solve this "puzzle"?

Outlining a solution to Game 1.

One possible solution method:

- Let's number the positions (rather than the pieces) in the game from position 1 to position n (left to right). First, find out on which position the piece with the highest number "n" lies. Let's say it is at the k-th position. If $k = n$, then this piece is already at it's target position. Otherwise, move it gradually from position k to $k+1$, then from $k+1$ to $k+2$, etc., and finally from $n-1$ to n be doing the respective "swaps".
- Now piece "n" is at its target position. We can now just ignore it, and keep manipulating the "rest" of the puzzle; i.e., in the next step, move piece number "$n-1$" to its position, then "$n-2$', and so on.

Generalization: The notion of a permutation puzzle

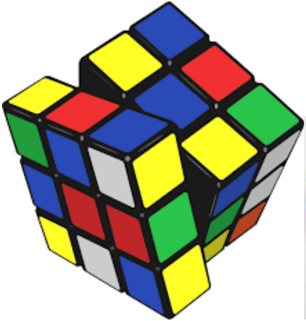
Instead of discussing the details of the above puzzle again, let's try to generalize the kind of game that we played above, and consider **any** game of the following form:

- There is a given number of positions labelled $1, \ldots, n$ on the game board, on which a total of n pieces (also numbered from 1 to n) are placed, but in an arbitrary order, say piece 1 on position x_1, piece 2 on position x_2, ..., and piece n on position x_n.
- There is a finite number of "legal moves". Every move consists of shuffling the pieces between the given positions in a prescribed way.
  (For example, "move the piece on position 1 to position 2, the one on position 2 to position 3, etc.)
- The legal moves always remain the same, in every moment of the game. A game consists in performing a sequence of legal moves, one after another.
- The game is considered solved, if all the pieces have been brought to the position of "their" respective number.

We call this kind of game a **permutation puzzle**.

Some famous examples

Here a some (a bit more difficult) famous examples of permutation puzzles:



Rubik's cube

Hungarian Rings

Pryraminx

Related concepts and results (students' point of view)

- Notion of a *permutation*: a 1-to-1 function from some set X to itself.
  In the "game situation": "Shuffling the pieces" between the positions 1, 2, $\dots, n$ in some prescribed way.
- Composing and inverting permutations ($f \circ g$, f^{-1}).
  In the game situation: "Performing several moves in a row"; "taking back a move".
- Compact notation for permutations
- Translating the solution of "Game 1" into a general theorem:

Theorem

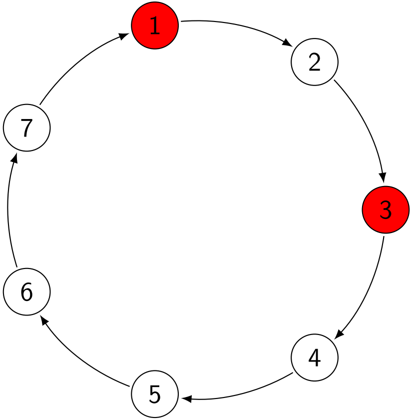
Any permutation of a set $\{1, \dots, n\}$ can be written as a composition of transpositions (="swaps").

Related concepts and resuts (teachers' points of view)

- Consider the set of all sequences of legal moves in a given permutation puzzle. Since composition (and inversion) of such sequences gives again a legal sequence, this set is in fact a **group** (in the algebraic sense), and more precisely a **subgroup** of the **symmetric group** on the set of all positions.
- The question whether any arbitrary initial constellation can be "solved" (i.e., brought into some prescribed order) is then exacty the question whether the above subgroup is the full symmetric group, or a smaller group.

Game 2: "Rotate and swap"

This time, let's try a slightly more challenging game.



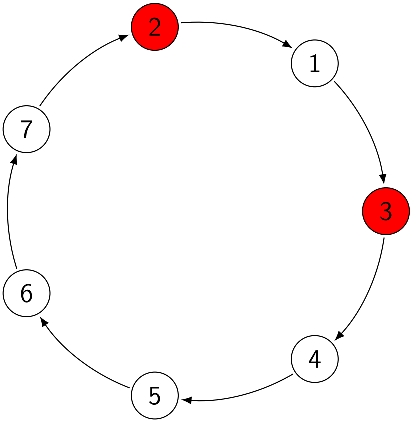
In this game, there are seven positions, and two legal moves:

- One consists of moving **each** piece clockwise to the next position.
- The second move consists in swapping the two pieces on the **marked** positions (without moving any other pieces).

Game 2: “Rotate and swap”

Activity (individual or group)

Task 1: Solve the following puzzle, in which only the pieces “1” and “2” should be swapped.



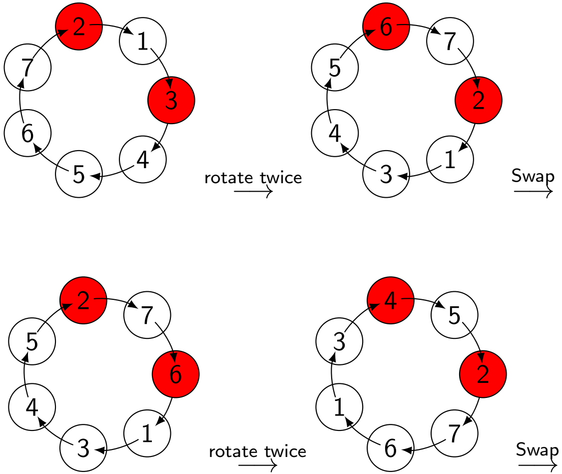
Game 2: "Rotate and swap"

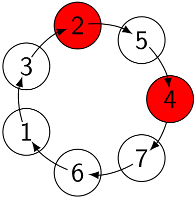
Additional questions:

- **Task 2**: Describe a strategy to perform an arbitrary swap "$a \leftrightarrow b$" (with any $1 \leq a < b \leq n$).
- **Task 3**: Use the results obtained so far to explain why **any** starting position in this puzzle can be solved.

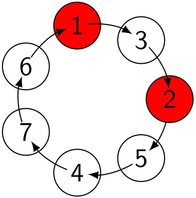
Related concepts and results

- "Conjugate $f^{-1} \circ g \circ f$ of one permutation g by another permutation f".
  - From students' perspective: Conjugate as a "trick" to perform an "essential" move g **after** having arranged pieces suitably through some sequence "f"; then "undoing" the undesired side effects of f by performing the inverse f^{-1}.
  - From teachers' perspective: Conjugation as a concept in group theory, e.g., related to notions such as "normal subgroups" etc.
- Inversion of a composition $f_1 \circ \cdots \circ f_r$ by performing the inverses in "reverse order"! ("Shoes and socks property").
- Application of the insights of "Game 1": Showing that any "Swap" can be performed in the game implies that any arbitrary permutation can be performed as well.
  - (From teachers' perspective: The symmetric group is "generated" by transpositions.)

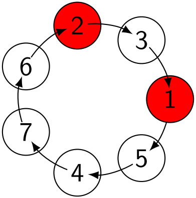

rotate twice
Swap
rotate twice
Swap



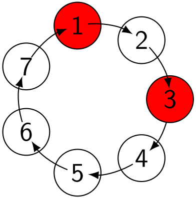
rotate twice $\longrightarrow$



Swap $\longrightarrow$



Undo all steps before the final swap $\longrightarrow$



Game 3: “Butterfly”

- Consider the following game pattern, consisting of five positions (arranged in two triangles), and legal moves $(1,2,3)$ and $(3,4,5)$ (i.e., rotating the left or the right triangle respectively).

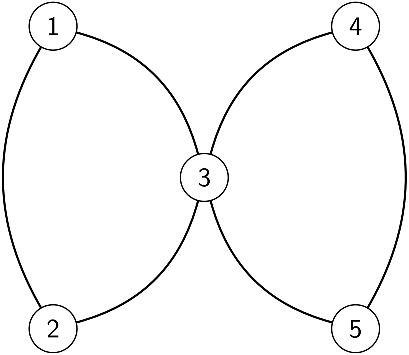


Figure: “Butterfly”: Starting position

- **Question**: Is it possible, through a sequence of legal moves, to interchange exactly the pieces on positions 1 and 2, without moving any other pieces?

Related concepts and results

- "Parity" of a permutation.
  - (Students' point of view: Parity as an invariant of a permutation, similar to the parity of an (odd or even) number. Parity argument to show that certain puzzle tasks are impossible, in much the same way as it is impossible to gain an odd integer by adding only even integers.)
  - (Teachers' point of view: Parity leading to the definition of "alternating group" as a subgroup of the symmetric group.)
- **Conclusion**: Above task is (provably!) impossible.

Game 4: Challenge

Activity (individual or group)

It's time to be creative! Make up your own "permutation puzzle" by designing a game board (in the form of a graph as before) and assigning a set of "legal moves".

Try to "analyze" your game. Can you show, using properties of permutations, that every position in your game can be solved? Or maybe can you show that there are positions that **cannot** be solved?

1. Motivation
2. Overview of course contents (examples)
3. Outlook

Experience and future challenges

- Implementation of a similar (but more advanced) course for selected High School Students in Cheongju, 2020.
- Possibilty of a training course "Applied group theory" for (future) teachers, relating course contents with university level abstract algebra concepts.
- Possibility of systematic investigation of teachers' conception of discrete math contents, "before" and "after".